UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Electrical &
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ECE 204 *Numerical methods*

Approximating integrals using interpolating polynomials

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Introduction

- In this topic, we will
  - Review the definition of a Riemann sum
  - Find different formulas for approximating integrals over an interval
  - Look at formulas that evaluate the function to the left and to the right of an interval and formulas that only use historic data
    - In each case, we will see how the error is reduced as *h* gets smaller
  - We will then sub-divide the interval of integration into sub-intervals and apply these formulas on each sub-interval
    - These are composite integration formulas
  - Discuss the effect of discontinuities on our approximation

Review

- In the previous topic, we discussed estimating the derivative of a function by interpolating polynomials and differentiating these polynomials
- We will now estimate the integral
  - First using Riemann sums
  - Next by integrating interpolating polynomials
  - Finally, we will discuss integration over a longer interval

- To compare methods, we will use a consistent step size h
  - Thus, $t_k = t_0 + kh$

Review

- In approximating an integral, we are sampling points either in space or time
  - We will use these sampling points to estimate the integral on an interval of the function
  - You have already seen one approximation: the Riemann sum

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} f(a+kh)h \text{ where } h = \frac{b-a}{n}$$

  - While this is correct, it is not very efficient:

$$\int_a^b f(x)\,\mathrm{d}x = \sum_{k=1}^{n} f(a+kh)h + (b-a)h\frac{1}{2}f^{(1)}(\xi)$$

The proof that the error is as given is beyond the scope of this course.

Riemann sums

- Let's approximate $\int_0^5 e^{-x}\cos(x)\,dx$

n	Approximation	Error	Ratio	Time (s)
10	0.2673897629069273	0.2284		< 1 μs
100	0.4710708401029400	0.02474	0.1083	0.0000023
1000	0.4933206302108438	0.002493	0.1008	0.000016
10000	0.4955642578982589	0.0002495	0.1001	0.0001911
100000	0.4957888071367516	0.00002495	0.1000	0.001600
1000000	0.4958112639253148	0.000002495	0.1000	0.01599
10000000	0.4958135096226953	0.0000002495	0.1000	0.1595
100000000	0.4958137341926944	0.00000002495	0.1000	1.579
1 billion	0.4958137566491973	0.000000002496	0.1000	15.77
10 billion	0.4958137588977893	0.0000000002471	0.0990	157.5
100 billion	0.4958137579192325	0.000000001226	4.959	1587
	0.4958137591449437			

Trapezoidal rule

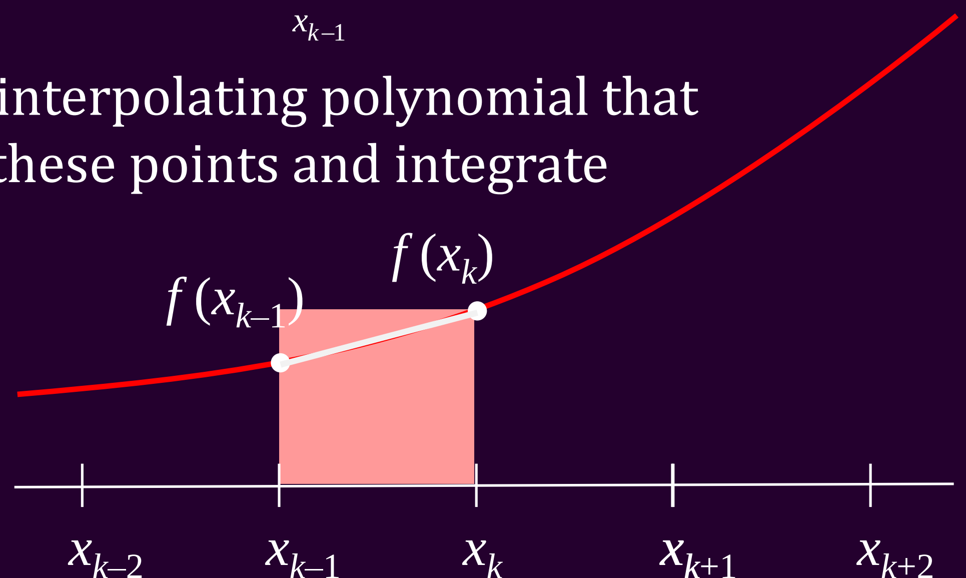
- In calculus, you learned that the integral is the area under a curve
  - Suppose we have the points

$$(x_{k-1}, f(x_{k-1})), (x_k, f(x_k))$$

  - We'd like to approximate

$$\int_{x_{k-1}}^{x_k} f(x)\,\mathrm{d}x$$

  - We can find the interpolating polynomial that passes through these points and integrate that polynomial



Trapezoidal rule

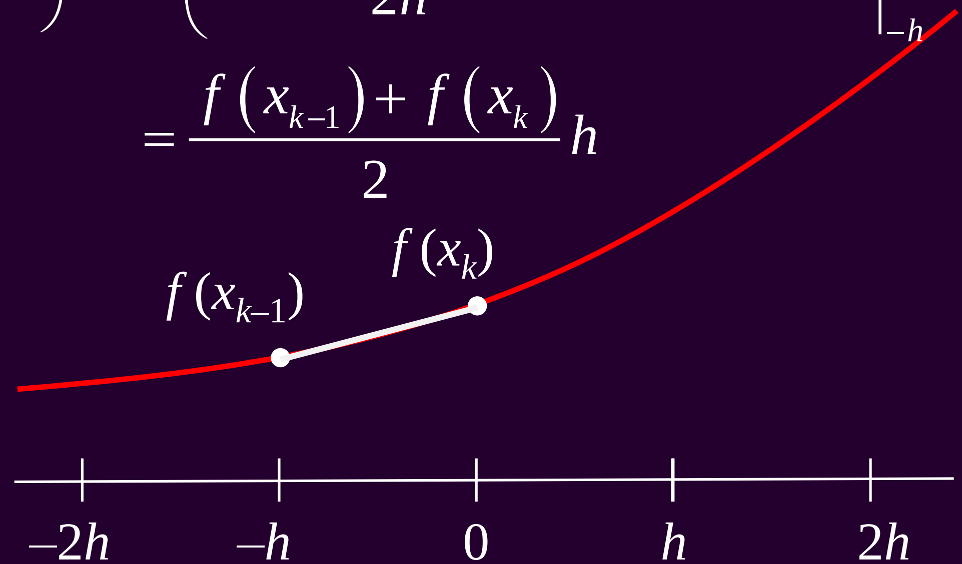
- As before, we will shift to zero, but not scale:

$$(-h, f(x_{k-1})), (0, f(x_k))$$

  – The interpolating line is $\frac{f(x_k) - f(x_{k-1})}{h} x + f(x_k)$ and thus the integral is

$$\int_{-h}^{0} \left(\frac{f(x_k) - f(x_{k-1})}{h} x + f(x_k) \right) \mathrm{d}x = \left(\frac{f(x_k) - f(x_{k-1})}{2h} x^2 + f(x_k) x \right|_{-h}^{0}$$

$$= \frac{f(x_{k-1}) + f(x_k)}{2} h$$



Trapezoidal rule

- Okay, so we now have this approximation:

$$\frac{f(x_{k-1}) + f(x_k)}{2} h$$

  – This is the same as

$$\frac{f(x_k - h) + f(x_k)}{2} h$$

  – What is the error?

$$\int_{x_{k-1}}^{x_k} f(x)\,\mathrm{d}x = \frac{f(x_{k-1}) + f(x_k)}{2} h - \frac{1}{12} f^{(2)}(\xi) h^3$$

The proof that the error is as given is beyond the scope of this course.

Trapezoidal rule

- Let's approximate $\int_{1}^{1+h} e^{-x} \cos(x)\,\mathrm{d}x$

n	$h = 2^{-n}$	Exact	Approximation	Error	$-\frac{1}{12} f^{(2)}(1) h^3$	Ratio
1	0.5	0.047996923832476	0.053637428370745	–0.005641	–0.006449	
2	0.25	0.035376755450843	0.036138434966243	–0.0007617	–0.0008061	0.1350
3	0.125	0.021073557644694	0.021171789399505	–0.00009823	–0.0001008	0.1290
4	0.0625	0.011455102718756	0.011467548714968	–0.00001245	–0.00001260	0.1267
5	0.03125	0.005966374906433	0.005967940313323	–0.000001565	–0.000001575	0.1258
6	0.015625	0.003044062031489	0.003044258284865	–0.0000001963	–0.0000001968	0.1254
7	0.0078125	0.001537396527721	0.001537421094646	–0.00000002457	–0.00000002460	0.1252
8	0.00390625	0.000772558047289	0.000772561120332	–0.000000003073	–0.000000003075	0.1251
9	0.001953125	0.000387246273062	0.000387246657327	–0.0000000003843	–0.0000000003844	0.1250
10	0.0009765625	0.000193865237008	0.000193865285050	–0.00000000004804	–0.00000000004805	0.1250

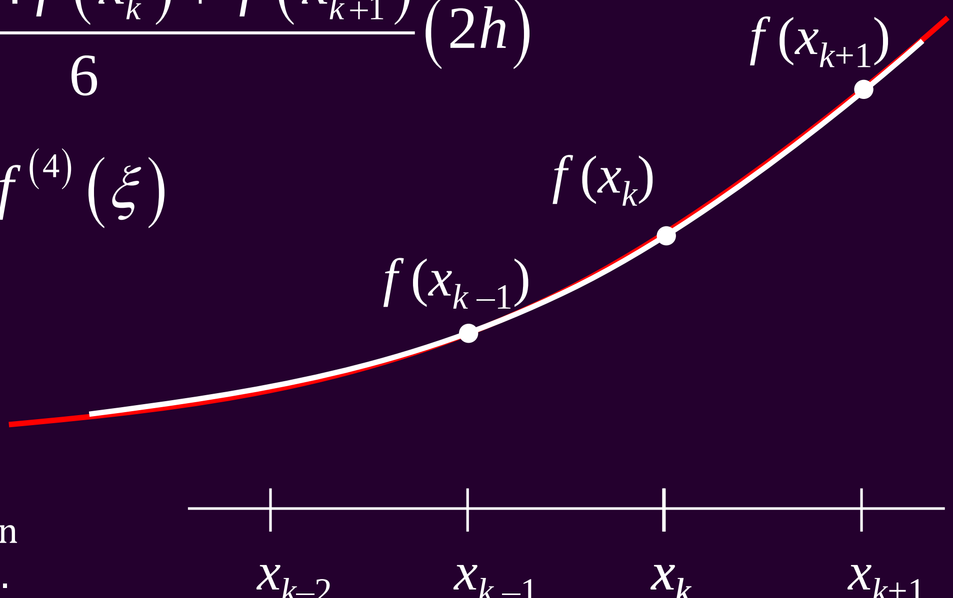
Simpson's rule

- Can we find a better estimate of the integral?
  - Suppose we find the interpolating quadratic polynomial between these three points:

$$(x_{k-1}, f(x_{k-1})), (x_k, f(x_k)), (x_{k+1}, f(x_{k+1}))$$

  - Integrate that quadratic from x_{k-1} to x_{k+1}

$$\int_{x_{k-1}}^{x_{k+1}} f(x)\,\mathrm{d}x = \frac{f(x_{k-1}) + 4f(x_k) + f(x_{k+1})}{6}(2h) - \frac{1}{90}h^5 f^{(4)}(\xi)$$



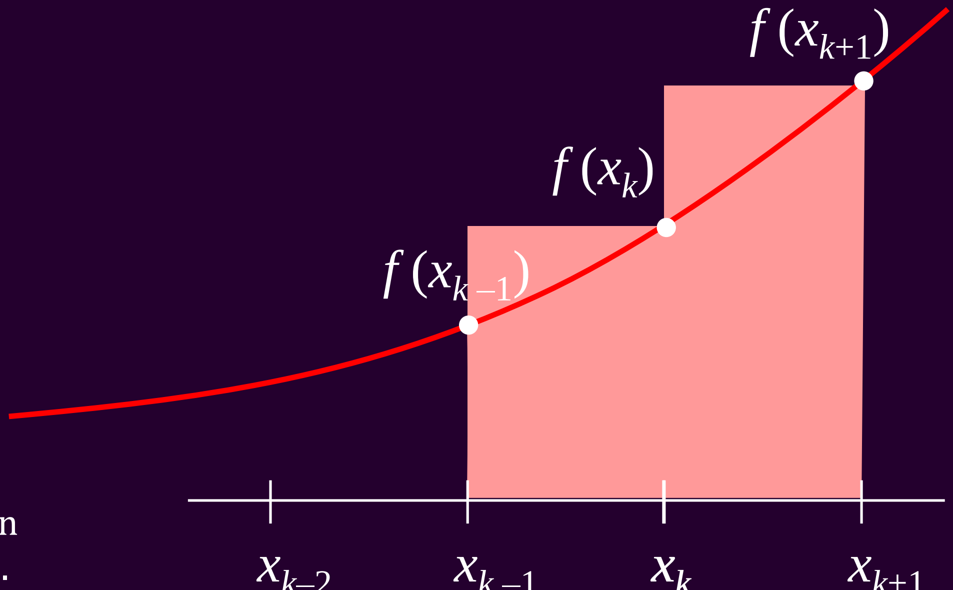
The proof that the error is as given is beyond the scope of this course.

Comparisons

$$\int_{x_{k-1}}^{x_{k+1}} f(x)\,\mathrm{d}x = \frac{f(x_k)+f(x_{k+1})}{2}(2h) + f^{(1)}(\xi)h^2$$

$$= \frac{f(x_{k-1})+2f(x_k)+f(x_{k+1})}{4}(2h) - \frac{1}{6}f^{(2)}(\xi)h^3$$

$$= \frac{f(x_{k-1})+4f(x_k)+f(x_{k+1})}{6}(2h) - \frac{1}{90}f^{(4)}(\xi)h^5$$



The proof that the error is as given is beyond the scope of this course.

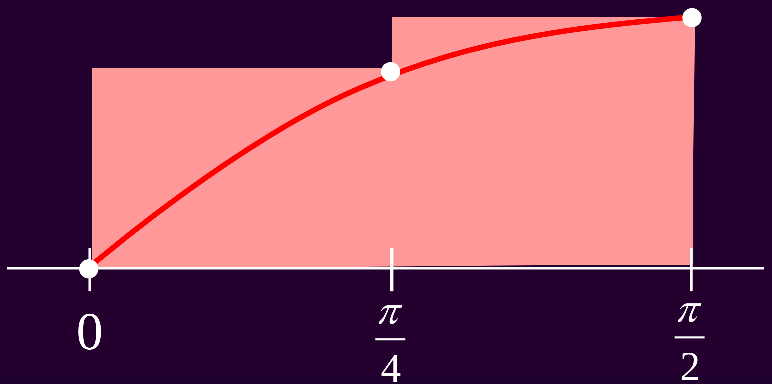
Comparisons

$$\int_0^{\frac{\pi}{2}} \sin(x)\,dx = 1$$

$$\frac{\sin\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{2}\right)}{2}\frac{\pi}{2} = 1.340758530667244$$

$$\frac{\sin(0)+2\sin\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{2}\right)}{4}\frac{\pi}{2} = 0.9480594489685198$$

$$\frac{\sin(0)+4\sin\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{2}\right)}{6}\frac{\pi}{2} = 1.002279877492211$$

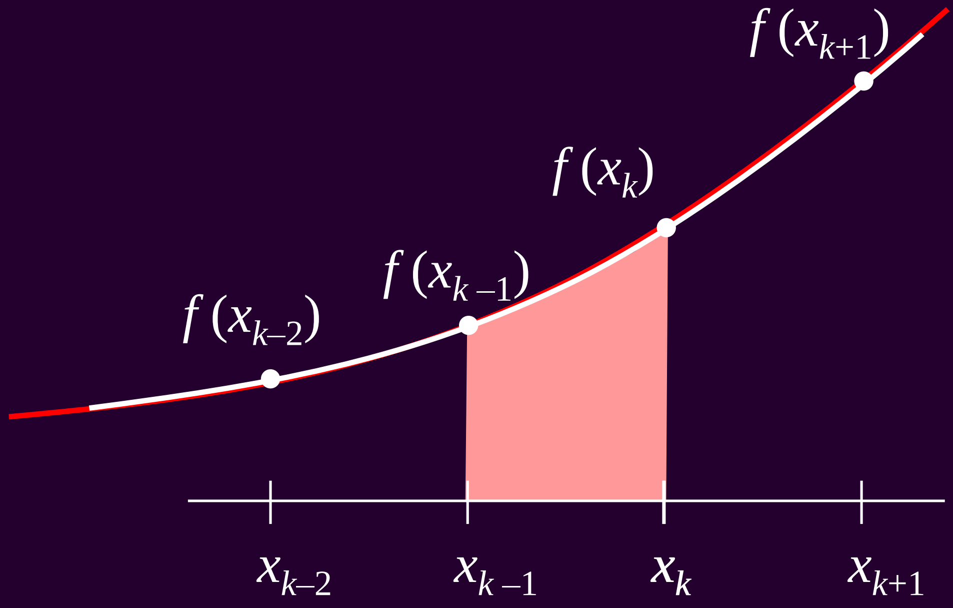


Centered approximations

- What can we do with one interval?
  - Suppose we find the interpolating cubic polynomial between these four points:

$$(x_{k-2}, f(x_{k-2})), (x_{k-1}, f(x_{k-1})), (x_k, f(x_k)), (x_{k+1}, f(x_{k+1}))$$

  - Integrate that cubic from x_{k-1} to x_k

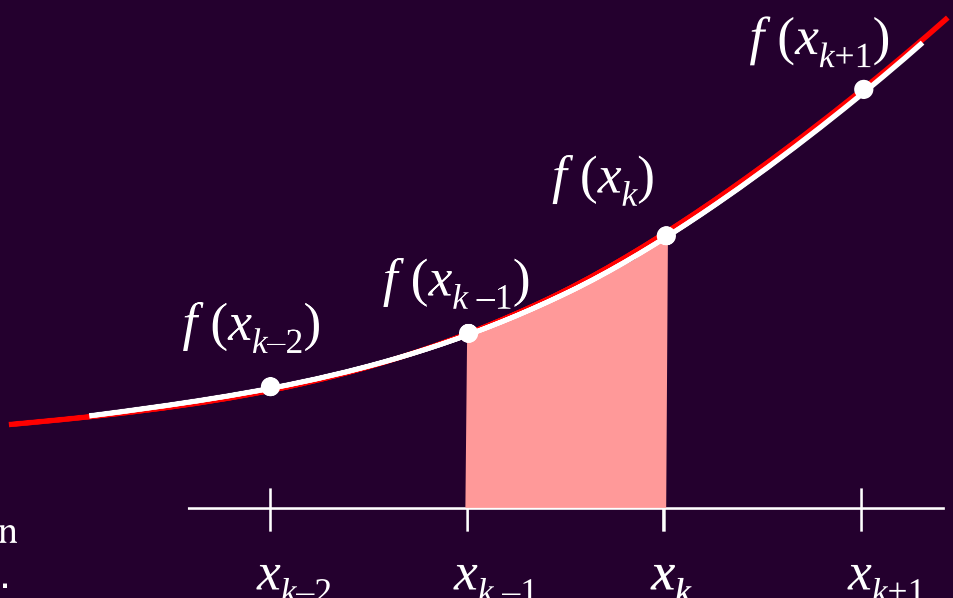


Centered approximations

- This gives us the approximation:

$$\int_{x_{k-1}}^{x_k} f(x)\,dx$$

$$= \frac{-f(x_{k-2}) + 13f(x_{k-1}) + 13f(x_k) - f(x_{k+1})}{24}h + \frac{11}{720}f^{(4)}(x_k)h^5 + O(h^6)$$



The proof that the error is as given is beyond the scope of this course.

O(h^5) centered formula

- Let's approximate $\int_1^{1+h} e^{-x}\cos(x)\,\mathrm{d}x$

n	$h = 2^{-n}$	Exact	Approximation	Error	$\frac{11}{720} f^{(4)}(1)h^5$	Ratio
1	0.5	0.04799692383247647	0.04819135198031643	-1.944×10^{-4}	-3.796×10^{-4}	
2	0.25	0.03537675545084305	0.03538529463212308	-8.539×10^{-6}	-1.186×10^{-5}	0.0439
3	0.125	0.02107355764469405	0.02107387281672892	-3.152×10^{-7}	-3.707×10^{-7}	0.0369
4	0.0625	0.01145510271875572	0.01145511340645540	-1.069×10^{-8}	-1.158×10^{-8}	0.0339
5	0.03125	0.005966374906432655	0.005966375254200959	-3.478×10^{-10}	-3.620×10^{-10}	0.0325
6	0.015625	0.003044062031488833	0.003044062042577209	-1.109×10^{-11}	-1.131×10^{-11}	0.0319
7	0.0078125	0.001537396527720683	0.001537396528070679	-3.500×10^{-13}	-3.535×10^{-13}	0.0316
8	0.00390625	0.0007725580472894567	0.0007725580473004350	-1.098×10^{-14}	-1.105×10^{-14}	0.0314
9	0.001953125	0.0003872462730616574	0.0003872462730619874	-3.300×10^{-16}	-3.452×10^{-16}	0.0301
10	0.0009765625	0.0001938652370081922	0.0001938652370082053	-1.301×10^{-17}	-1.079×10^{-17}	0.0394

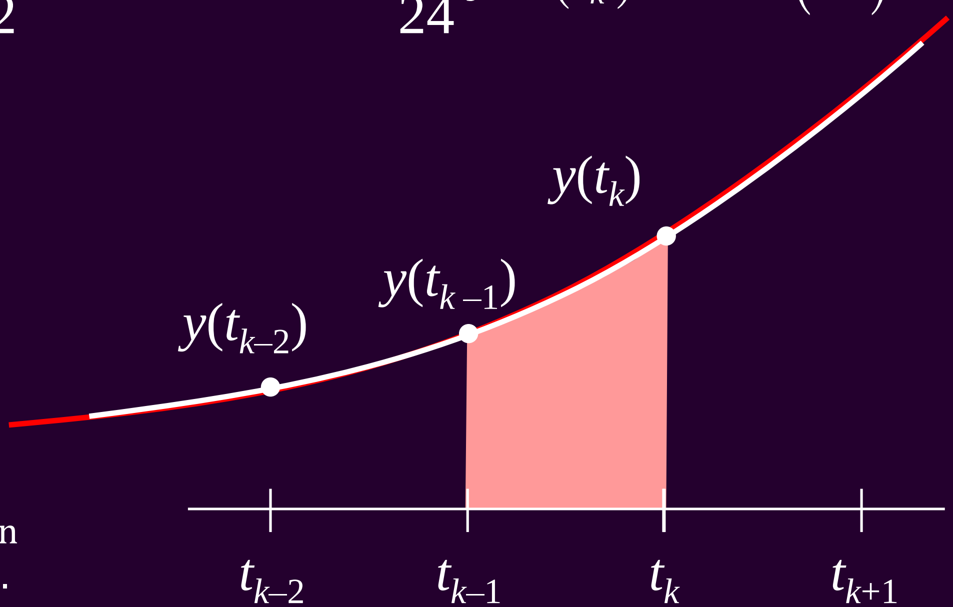
$$\int_{x_{k-1}}^{x_k} f(x)\,\mathrm{d}x = \frac{-f(x_{k-2}) + 13f(x_{k-1}) + 13f(x_k) - f(x_{k+1})}{24}h + \frac{11}{720}f^{(4)}(x_k)h^5 + O(h^6)$$

Backward approximations

- What happens if we don't have information from beyond the current time?
  - Integrate the interpolating polynomial on the points

$$(t_{k-2}, y(t_{k-2})), (t_{k-1}, y(t_{k-1})), (t_k, y(t_k))$$

$$\int_{t_{k-1}}^{t_k} y(t)\,\mathrm{d}t = \frac{-y(t_{k-2}) + 8y(t_{k-1}) + 5y(t_k)}{12}h - \frac{1}{24}y^{(3)}(t_k)h^4 + O(h^5)$$



The proof that the error is as given is beyond the scope of this course.

O(h^4) backward formula

- Let's approximate $\int_1^{1+h} e^{-t}\cos(t)\,dt$

n	$h = 2^{-n}$	Exact	Approximation	Error	$-\frac{1}{24}y^{(3)}(1)h^4$	Ratio
1	0.5	0.04799692383247647	0.04736525700993603	6.317×10^{-4}	5.771×10^{-4}	
2	0.25	0.03537675545084305	0.03533771627125132	3.904×10^{-5}	3.607×10^{-5}	0.0618
3	0.125	0.02107355764469405	0.02107118998018845	2.368×10^{-6}	2.254×10^{-6}	0.0606
4	0.0625	0.01145510271875572	0.01145495795902083	1.448×10^{-7}	1.409×10^{-7}	0.0611
5	0.03125	0.005966374906432655	0.005966365974938867	8.931×10^{-9}	8.805×10^{-9}	0.0617
6	0.015625	0.003044062031488833	0.003044061477138604	5.544×10^{-10}	5.503×10^{-10}	0.0621
7	0.0078125	0.001537396527720683	0.001537396493198497	3.452×10^{-11}	3.439×10^{-11}	0.0623
8	0.00390625	0.0007725580472894567	0.0007725580451357632	2.154×10^{-12}	2.150×10^{-12}	0.0624
9	0.001953125	0.0003872462730616574	0.0003872462729271626	1.345×10^{-13}	1.344×10^{-13}	0.0624
10	0.0009765625	0.0001938652370081922	0.0001938652369997934	8.399×10^{-15}	8.397×10^{-15}	0.0624

$$\int_{t_{k-1}}^{t_k} y(t)\,dt = \frac{-y(t_{k-2}) + 8y(t_{k-1}) + 5y(t_k)}{12}h - \frac{1}{24}y^{(3)}(t_k)h^4 + O(h^5)$$

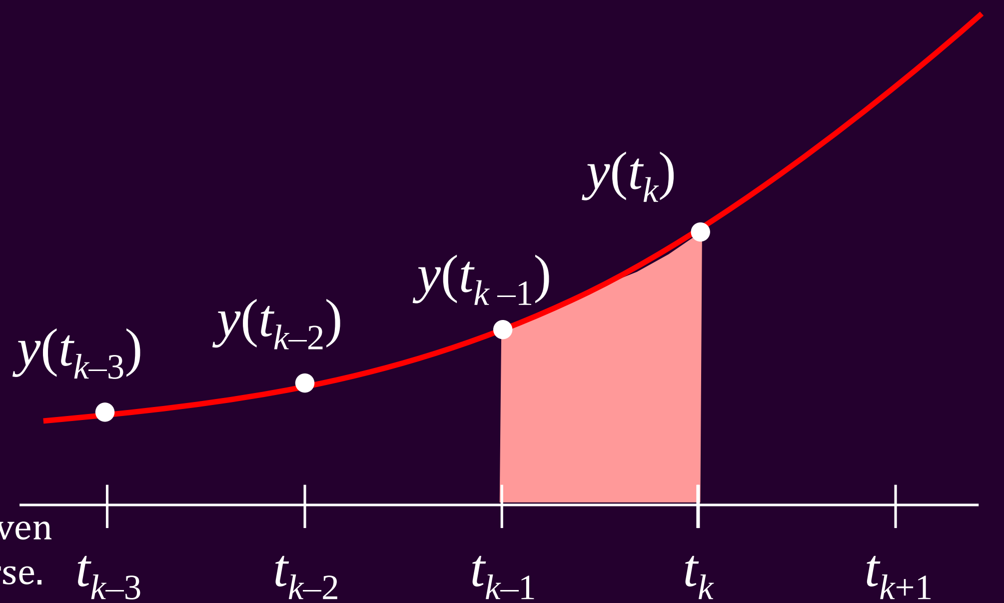
The proof that the error is as given is beyond the scope of this course.

Backward approximations

- Can we get a more accurate approximation?
  - Integrate the interpolating polynomial on the points

$$(t_{k-3}, y(t_{k-3})), (t_{k-2}, y(t_{k-2})), (t_{k-1}, y(t_{k-1})), (t_k, y(t_k))$$

$$\int_{t_{k-1}}^{t_k} y(t)\,dt = \frac{y(t_{k-3}) - 5y(t_{k-2}) + 19y(t_{k-1}) + 9y(t_k)}{24}h + \frac{19}{720}y^{(4)}(t_k)h^5 + O(h^6)$$



The proof that the error is as given is beyond the scope of this course.

O(h^5) backward formula

- Let's approximate $\int_1^{1+h} e^{-t}\cos(t)\,dt$

n	$h = 2^{-n}$	Exact	Approximation	Error	$\frac{19}{720}y^{(4)}(1)h^5$	Ratio
1	0.5	0.047996923832476 47	0.0470251015360956 1	9.718×10^{-4}	6.557×10^{-4}	
2	0.25	0.0353767554508430 5	0.0353519000823417 4	2.486×10^{-5}	2.049×10^{-5}	0.02558
3	0.125	0.0210735576446940 5	0.0210728536518439 0	7.040×10^{-7}	6.403×10^{-7}	0.02832
4	0.0625	0.0114551027187557 2	0.0114550817493904 2	2.107×10^{-8}	2.001×10^{-8}	0.02979
5	0.03125	0.00596637490643265 5	0.00596637426641853 7	6.400×10^{-10}	6.253×10^{-10}	0.03052
6	0.015625	0.00304406203148883 3	0.00304406201172073 7	1.977×10^{-11}	1.954×10^{-11}	0.03089
7	0.0078125	0.00153739652772068 3	0.00153739652710650 6	6.142×10^{-13}	6.106×10^{-13}	0.03107
8	0.00390625	0.000772558047289456 7	0.000772558047270305 0	1.915×10^{-14}	1.908×10^{-14}	0.03118
9	0.001953125	0.000387246273061657 4	0.000387246273061045 8	6.117×10^{-16}	5.963×10^{-16}	0.03194
10	0.0009765625	0.000193865237008192 2	0.000193865237008175 8	1.640×10^{-17}	1.863×10^{-17}	0.02681

$$\int_{t_{k-1}}^{t_k} y(t)\,dt = \frac{y(t_{k-3}) - 5y(t_{k-2}) + 19y(t_{k-1}) + 9y(t_k)}{24}h + \frac{19}{720}y^{(4)}(t_k)h^5 + O(h^6)$$

The proof that the error is as given is beyond the scope of this course.

The one-step formulas

- We have four formulas for approximating $\int_{x_{k-1}}^{x_k} f(x)\,\mathrm{d}x$ or $\int_{t_{k-1}}^{t_k} y(t)\,\mathrm{d}t$:

$$\frac{f(x_{k-1}) + f(x_k)}{2}h - \frac{1}{12}f^{(2)}(\xi)h^3$$

$$\frac{-f(x_{k-2}) + 13f(x_{k-1}) + 13f(x_k) - f(x_{k+1})}{24}h + \frac{11}{720}f^{(4)}(\xi)h^5$$

$$\frac{y(t_{k-1}) + y(t_k)}{2}h - \frac{1}{12}y^{(2)}(\tau)h^3$$

$$\frac{-y(t_{k-2}) + 8y(t_{k-1}) + 5y(t_k)}{12}h + \frac{1}{24}y^{(3)}(t_k)h^4 + O(h^5)$$

$$\frac{y(t_{k-3}) - 5y(t_{k-2}) + 19y(t_{k-1}) + 9y(t_k)}{24}h + \frac{19}{270}y^{(4)}(t_k)h^5 + O(h^6)$$

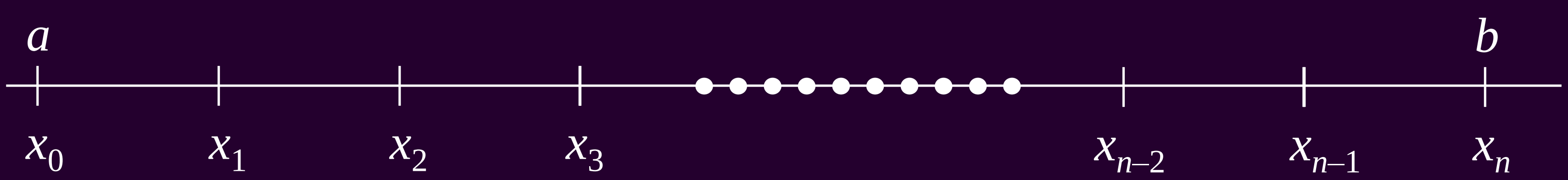
Integrating over a fixed interval

- The only way to reduce the error is to reduce *h*
  - That is, reduce the interval over which we calculate the integral
- Instead,
  - Divide the interval into *n* *sub-intervals*

$$h = \frac{b-a}{n} \qquad x_k = a + kh$$

  - The complete integral equals the sum of the integrals over each sub-interval:

$$\int_a^b f(x)\,\mathrm{d}x = \sum_{k=1}^{n} \left(\int_{x_{k-1}}^{x_k} f(x)\,\mathrm{d}x \right) \qquad \int_{t_0}^{T} y(t)\,\mathrm{d}t = \sum_{k=1}^{n} \left(\int_{t_{k-1}}^{t_k} y(t)\,\mathrm{d}t \right)$$



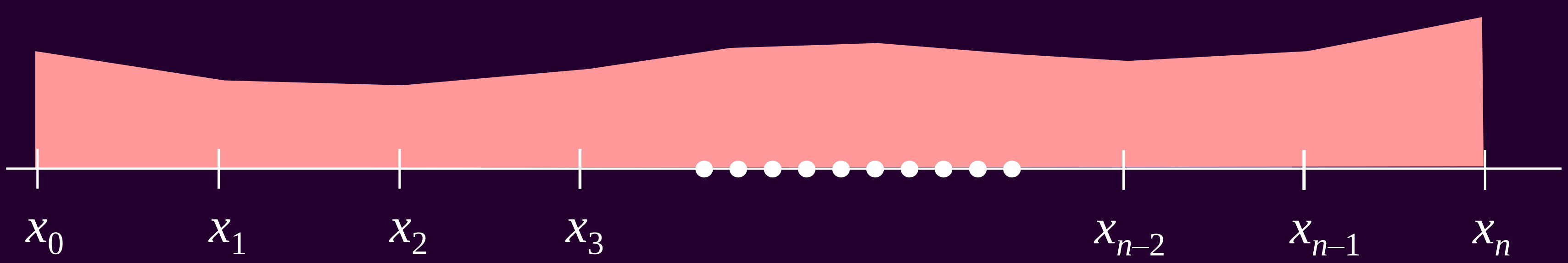
Composite trapezoidal rule

- Thus, we can approximate an integral as follows:

$$\int_a^b y(x)\,\mathrm{d}x = \sum_{k=1}^{n}\left(\int_{x_{k-1}}^{x_k} f(x)\,\mathrm{d}x\right)$$

$$= \sum_{k=1}^{n}\left(\frac{f(x_{k-1}) + f(x_k)}{2}h - \frac{1}{12}f^{(2)}(\xi_k)h^3\right)$$

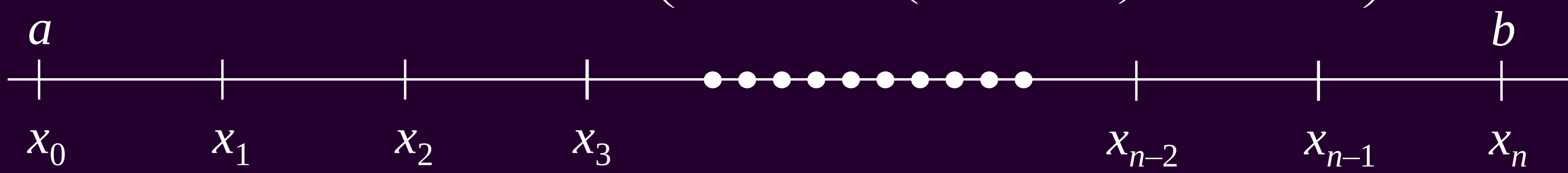
$$= h\sum_{k=1}^{n}\frac{f(x_{k-1}) + f(x_k)}{2} - \frac{h^3}{12}\sum_{k=1}^{n} f^{(2)}(\xi_k)$$



Composite trapezoidal rule

- The formula is straight-forward:

$$h\sum_{k=1}^{n}\frac{f\left(x_{k-1}\right)+f\left(x_k\right)}{2}=h\left(\frac{1}{2}f\left(x_0\right)+\left(\sum_{k=1}^{n-1}f\left(x_k\right)\right)+\frac{1}{2}f\left(x_n\right)\right)$$



- How about the error?

$$-\frac{h^3}{12}\sum_{k=1}^{n}f^{(2)}\left(\xi_k\right)=-\frac{nh^3}{12}\sum_{k=1}^{n}\frac{1}{n}f^{(2)}\left(\xi_k\right)$$

$$=-\frac{\left(b-a\right)h^2}{12}f^{(2)}\left(\xi\right)$$

$$h=\frac{b-a}{n}$$

$$nh=b-a$$

Implementation

- Here is an implementation:

```
double int_O_h2( double f( double x ), double a,
                 double b, unsigned int n ) {
    double h{ (b - a)/n };

    double sum{ (f(a) + f(b))/2.0 };

    for ( int k{1}; k <= n - 1; ++k ) {
        sum += f(a + k*h);
    }

    return sum*h;
}
```

C/C++ Code is provided to demonstrate the straight-forward nature of these algorithms and not required for the examination

Composite trapezoidal rule

- Let's approximate $\int_0^5 e^{-x}\cos(x)\,dx$

k	$n = 2^k$	Approximation	Error	$-5\cdot\frac{1}{12}\overline{f^{(2)}_{[0,5]}}h^2$	Ratio
1	2	1.0879844444514831	–0.5922	–0.5232	
2	4	0.63279680095 47279	–0.1370	–0.1308	0.2313
3	8	0.5289261874144371	–0.03311	–0.03270	0.2417
4	16	0.5040149518112748	–0.008201	–0.008175	0.2477
5	32	0.4978591609946217	–0.002045	–0.002043	0.2494
6	64	0.4963248022194289	–0.0005110	–0.0005109	0.2498
7	128	0.4959415006805423	–0.0001277	–0.0001277	0.2500
8	256	0.4958456933264472	–0.00003193	–0.00003193	0.2500
9	512	0.4958217426151648	–0.00007983	–0.000007983	0.2500
10	1024	0.4958157550078016	–0.000001996	–0.000001996	0.2500
		0.4958137591449437			

Composite O(h^5) centered formula

- Thus, we can calculate the following:

$$\int_a^b y(x)\,\mathrm{d}x = \sum_{k=1}^{n}\left(\int_{x_{k-1}}^{x_k} f(x)\,\mathrm{d}x\right)$$

$$= \sum_{k=1}^{n}\left(\frac{-f(x_{k-2})+13f(x_{k-1})+13f(x_k)-f(x_{k+1})}{24}h + \frac{11}{720}f^{(4)}(x_k)h^5 + O(h^6)\right)$$

$$= \frac{h}{24}\sum_{k=1}^{n}\left(-f(x_{k-2})+13f(x_{k-1})+13f(x_k)-f(x_{k+1})\right) + h^5\frac{11}{720}\sum_{k=1}^{n}\left(f^{(4)}(x_k)+O(h)\right)$$

Composite O(h^5) centered formula

- With a little work, the approximation can be seen to be:

$$\frac{h}{24}\sum_{k=1}^{n}\left(-f\left(x_{k-2}\right)+13f\left(x_{k-1}\right)+13f\left(x_k\right)-f\left(x_{k+1}\right)\right)$$

$$=h\left(-\frac{1}{24}f\left(x_{-1}\right)+\frac{1}{2}f\left(x_0\right)+\frac{25}{24}f\left(x_1\right)+\left(\sum_{k=2}^{n-2}f\left(x_k\right)\right)\right.$$

$$\left.+\frac{25}{24}f\left(x_{n-1}\right)+\frac{1}{2}f\left(x_n\right)-\frac{1}{24}f\left(x_{n+1}\right)\right)$$

- How about the error?

$$h^5\frac{11}{720}\sum_{k=1}^{n}f^{(4)}\left(\xi_k\right)=nh^5\frac{11}{720}\sum_{k=1}^{n}\frac{1}{n}\left(f^{(4)}\left(x_k\right)+O\left(h\right)\right)$$

$$=\left(b-a\right)h^4\frac{11}{720}f^{(4)}\left(\xi\right)+O\left(h^5\right)$$

Implementation

- Here is an implementation:

```
double int_O_h4( double f( double x ), double a,
                 double b, unsigned int n ) {
    double h{ (b - a)/n };

    double sum{ -(f(a - h) + f(b + h))/24.0
               + (f(a)     + f(b)    )/2.0
               + (f(a + h) + f(b - h))*(25.0/24.0) };

    for ( int k{2}; k <= n - 2; ++k ) {
        sum += f(a + k*h);
    }

    return sum*h;
}
```

C/C++ Code is provided to demonstrate the straight-forward nature of these algorithms and not required for the examination

Composite O(h^5) centered formula

- Let's approximate $\int_0^5 e^{-x} \cos(x)\,dx$

k	$n = 2^k$	Approximation	Error	$5 \cdot \frac{11}{720} \overline{f^{(4)}_{[0,5]}} h^4$	Ratio
1	2	1.9265184916406260	–1.431	–1.184	
2	4	0.5790744616202452	–0.08326	–0.07397	0.05820
3	8	0.5005922957727700	–0.004779	–0.004623	0.05739
4	16	0.4961051823558393	–0.0002914	–0.0002890	0.06099
5	32	0.4958318578145426	–0.00001810	–0.00001806	0.06210
6	64	0.4958148885018988	–0.000001129	–0.000001129	0.06240
7	128	0.4958138297014402	–0.00000007056	–0.00000007055	0.06247
8	256	0.4958137635542820	–0.000000004409	–0.000000004409	0.06249
9	512	0.4958137594205204	–0.0000000002756	–0.0000000002756	0.06250
10	1024	0.4958137591621671	–0.00000000001722	–0.00000000001722	0.06250
		0.4958137591449437			

Composite backward formulas

- The backward formulas would be used when you are dynamically calculating the integral in real time
  - An appropriate variable is initialized to zero
  - With each time step, the integral over this time step is added to this variable

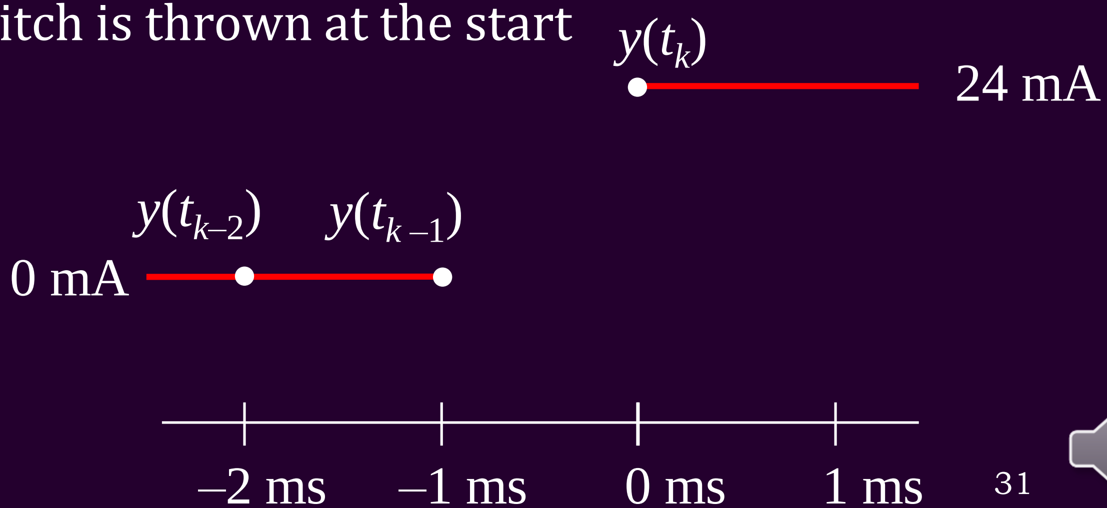
$$\sum_{k=1}^{n} \frac{y(t_{k-1}) + y(t_k)}{2} h - (t_n - t_0) \frac{1}{12} y^{(2)}(\tau) h^2$$

$$\sum_{k=1}^{n} \frac{-y(t_{k-2}) + 8y(t_{k-1}) + 5y(t_k)}{12} h + (t_n - t_0) \frac{1}{24} y^{(3)}(\tau) h^3 + O(h^4)$$

$$\sum_{k=1}^{n} \frac{y(t_{k-3}) - 5y(t_{k-2}) + 19y(t_{k-1}) + 9y(t_k)}{24} h + (t_n - t_0) \frac{19}{270} y^{(4)}(\tau) h^4 + O(h^5)$$

Discontinuities

- Suppose you are measuring current or voltage and a switch is flipped
  - There could be a discontinuity in either of these values
    - For example, the switch could be turned on at any point in the last millisecond
  - The total charge could be anywhere between
    - 0 μC if the switch is thrown at the last possible moment
    - 24 μC if the switch is thrown at the start



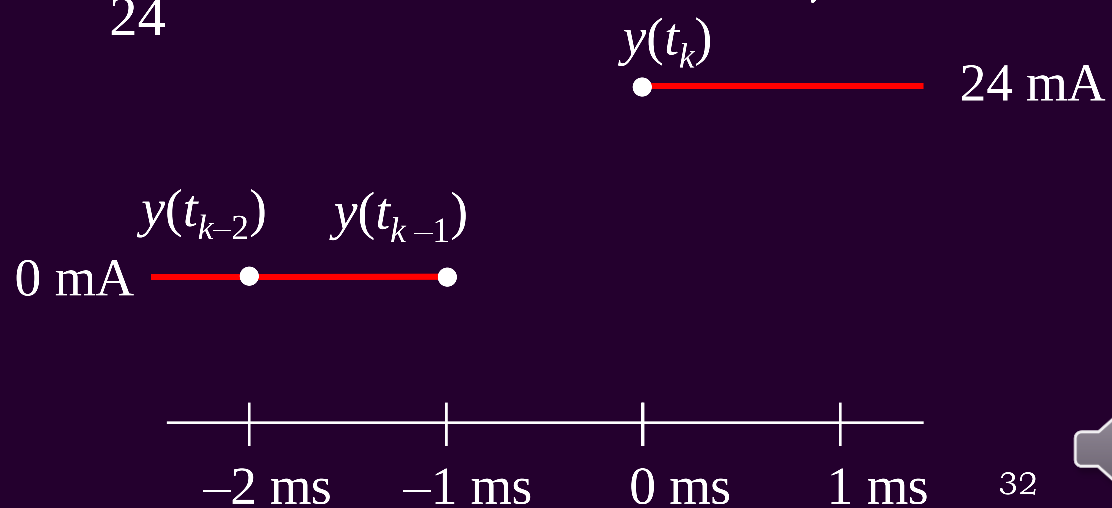
Discontinuities

- Thus, the answer is somewhere between 0 μC and 24 μC
  - Let's consider our three algorithms:

$$\frac{0\text{ mA} + 24\text{ mA}}{2} \cdot 1\text{ ms} = 12\ \mu\text{C}$$

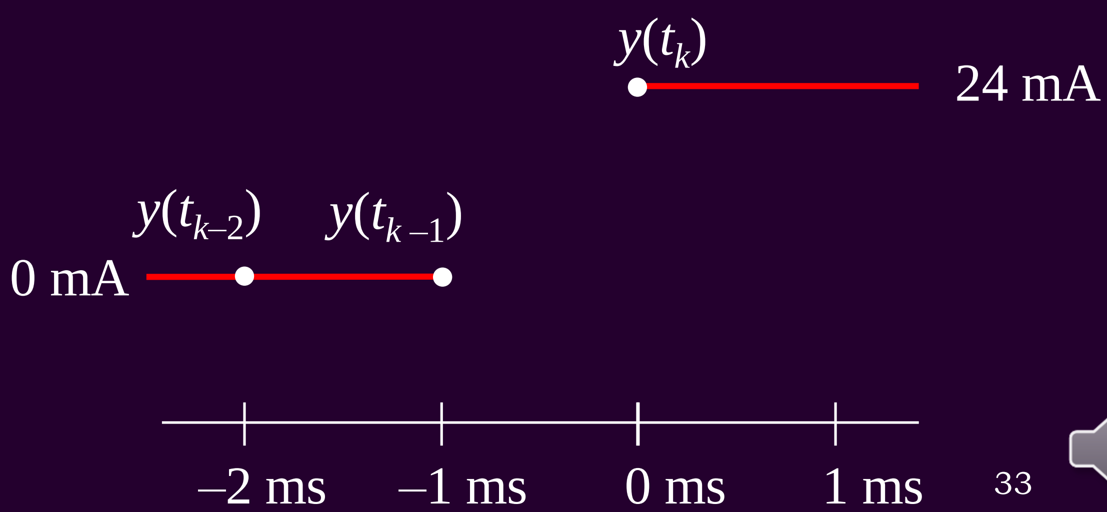
$$\frac{-0\text{ mA} + 8 \cdot 0\text{ mA} + 5 \cdot 24\text{ mA}}{12} \cdot 1\text{ ms} = 10\ \mu\text{C}$$

$$\frac{0\text{ mA} - 5 \cdot 0\text{ mA} + 19 \cdot 0\text{ mA} + 9 \cdot 24\text{ mA}}{24} \cdot 1\text{ ms} = 9\ \mu\text{C}$$



Discontinuities

- The *most* accurate (with still a huge error) is the trapezoidal rule
  - Use a more accurate technique until a discontinuity is detected
  - In this case, revert to the trapezoidal rule for one step



Summary

- Following this topic, you now
  - Are aware of numerous approximations the integral
  - Understand the trapezoidal rule and a centered O(h^5) rule
  - Understand that we can also approximate the integral over an interval using only historical data:
    - The trapezoidal rule on the last interval, and two others
  - Are aware of how to compose these approximations to find good approximations for integrals over larger intervals
  - Have seen that these do actually work as promised
  - Understand that discontinuities in our function will result in large errors and that we should revert to the trapezoidal rule

References

[1] https://en.wikipedia.org/wiki/Trapezoidal_rule

Acknowledgments

Tazik Shahjahan for pointing out typos.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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